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# The Laplacian in Regions with Many Small Obstacles: Fluctuations Around the Limit Operator

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We consider the Laplacian $\Delta_m$ in $\mathbb{R}^3$ (or in a bounded region of $\mathbb{R}^3$) with Dirichlet boundary conditions on the surfaces of some identical (small) neighborhoods of $m$ randomly distributed points, in the limit when $m$ goes to infinity and their linear size decreases as $1/m$. We give here a stronger form of the result showing the convergence of the above operator to $\Delta - C(x)$, where $C(x)$ is the limit density of electrostatic capacity of the "obstacles." In particular results on the rate of convergence and on the fluctuations of $\Delta_m$ around the limit operator are givei..

**KEY WORDS:** Random walks with traps; random media; potential theory; method of images; central limit theorem.

## 1. INTRODUCTION, BASIC DEFINITIONS AND RESULTS

Let us introduce first some notation. We will denote by $\Omega$ an open region of $\mathbb{R}^3$ and by $\underline{w}^{(m)} \equiv \{w_1,..., w_m\} \in (\Omega)^m$ a set of $m$ points $w_i \in \Omega$. Let $B$ be a simply connected open neighborhood of the origin with $C^2$ boundary $\partial B$. $B_j^\kappa(\underline{w}^{(m)})$ will indicate the set $\{x \in \Omega \mid (x - w_j)\, m^\kappa \in B\}$, $\kappa > 0$, and $D^\kappa(\underline{w}^{(m)})$ the set $\Omega \setminus \bigcup_j \overline{B_j^\kappa}(\underline{w}^{(m)})$.

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A prototype problem in the theory of "effective media" can be roughly stated in the following way: which are the asymptotic properties of the solution of the Laplace equation (heat equation, wave equation, etc.) on $D^{\kappa}(w^{(m)})$, with Dirichlet boundary conditions on $D^{\kappa}(\underline{w}^{(m)})$, in the limit in which the number $m$ of the "obstacles"is going to infinity and the points are distributed in some smooth way in $\Omega$? In particular, does an equation exist for the solutions of the limit problem?

Scattering of light by (many) opaque obstacles, quantum scattering by (many) hard core centers, propagation of heat in a composite medium in which the sparse component is kept at zero temperature, etc. are examples of physical situations for which the above setting can be taken as a model.

Many authors (Refs. 1–14) using different techniques studied this problem during the last decade (with a notable exception, Ref. 1, going back to 1948 as quoted in Ref. 6). Even more general assumptions on the obstacles than the ones described above have been discussed. From Refs. 1–14 we know that in the limit of an infinite number of obstacles the effect of the Dirichlet boundary conditions on the obstacles consists in replacing $\Delta$ with $\Delta - C(x)$, where the nonnegative function $C(x)$ characterizes the limit density of electrostatic capacity of the excluded set. In the presence of a smooth distribution of obstacles, this implies, in particular, that the only nontrivial limit is obtained for $\kappa = 1$. We will consider only this case and write $B_j$, $D$ for $B_j^1$, $D^1$. (For the rate of convergence to the limit of the eigenvalues in the "totally opaque" case, $\kappa < 1$, see Ref. 13).

In particular in a recent paper[12] Ozawa treated the case in which the points are independently, randomly distributed using an "image-charge" technique. He was able to give a lower bound for the rate of convergence to the limit solutions. Exploiting further his method we want to characterize the fluctuations around the limit solutions.

The notation we will use is the following: $G^{\lambda}(x, y)$, $\lambda > 0$, will denote the integral kernel of $(-\Delta + \lambda)^{-1}$ where $\Delta$ is the Laplacian with Dirichlet boundary condition on $\partial\Omega$. We will always assume that $\partial\Omega$ is piecewise $C^2$ to be guaranteed of the existence of $G^{\lambda}(x, y)$. $G^{\lambda}$ will denote the associated operator on $L^2(\Omega)$.

$G_m^{\lambda}(x, y; w^{(m)})$ will denote the integral kernel of $(-\Delta_m + \lambda)^{-1}$, where $\Delta_m$ is the Laplacian on $D(w^{(m)})$ with Dirichlet boundary condition on $\partial D(\underline{w}^{(m)})$. $G_m^{\lambda}$ will indicate the corresponding operator on $L^2(D(\underline{w}^{(m)}))$.

$\mathsf{G}^{\lambda}(\underline{w}^{(m)})$ will denote the $m \times m$ matrix $[\mathsf{G}^{\lambda}(\underline{w}^{(m)})]_{ij} = G^{\lambda}(w_i, w_j)$ for $i \neq j$, $[\mathsf{G}^{\lambda}(\underline{w}^{(m)})]_{ii} = 0$, $\forall i$.

$\mathbf{G}_x^{\lambda}(\underline{w}^{(m)})$, $x \in D(\underline{w}^{(m)})$, will denote the vector in $\mathbb{R}^m$ with components $[\mathbf{G}_x^{\lambda}]_i = G^{\lambda}(x, w_i)$ and $\mathbf{G}_f^{\lambda}(\underline{w}^{(m)})$, $f \in L^2(D(\underline{w}^{(m)}))$, the vector with components $[\mathbf{G}_f^{\lambda}(\underline{w}^{(m)})]_i = (G^{\lambda}f)(w_i)$.

$\|\cdot\|_p$ and $\|\cdot\|_p^{(m)}$ will indicate the norms of functions, respectively, in

$L^p(\mathbb{R}^3)$ and $L^p(D(\underline{w}^{(m)}))$ (for $p=2$ the same symbols will be used for operator norm) and $|||\cdot|||^{(m)}$ the norm of vectors and matrices in $\mathbb{R}^m$. Only real-valued functions on $\mathbb{R}^3$ will be considered.

Throughout the paper $C$ will indicate a positive real constant which may be different at each step in the proofs.

The case we are going to study is the one in which the $w_i$'s are independent, identically distributed random vectors in $\Omega$. The distribution is assumed to admit a continuous density $V(x)$.

As in Ref. 12 the techniques we are going to use are essentially non-probabilistic. We will characterize a set $W^{(m)}$ of configurations of points by some regularity conditions and we will prove our results for each element of $W^{(m)}$. It is easy to verify that the $\{V(x)\,dx\}^{\otimes m}$ measure of $W^{(m)}$ converges to 1 as $m$ goes to infinity.

More specifically for any positive $v<1/3$, we will define $W^{(m)}$ to be the set of those points $w^{(m)}$ in $((\Omega)^m, \{V(x)\,dx\}^{\otimes m})$ which satisfy the following assumptions:

$$\mathrm{A}_1: \qquad \min_{\substack{i\neq j\\ i,j=1,\ldots,m}} |w_i-w_j| \geqslant Cm^{-1+v}$$

$$\mathrm{A}_2: \qquad m^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{m} |w_i-w_j|^{-3+\xi} \leqslant C_\xi < \infty \qquad \text{for any} \quad \xi>0$$

Independence of the $w_i$'s, continuity of $V$ and the law of large numbers guarantee[12] that the set $W^{(m)}$ on which the assumptions $\mathrm{A}_1$ and $\mathrm{A}_2$ hold has a measure going to 1 as $m$ goes to infinity.

It should be noticed that the assumption $\mathrm{A}_1$ implies in particular that, independently of the initial size of the obstacles, they are not going to have intersections among each other for any configurations of points in $W^{(m)}$, for $m$ sufficiently large. We will always consider $m$ large enough to fulfil the above request.

In the following we will give details only for the case $\Omega\equiv\mathbb{R}^3$; the reader can verify at each step that the consideration of a domain $\Omega$ different from $\mathbb{R}^3$ introduces only minor changes and does not affect the proofs. When $\Omega\equiv\mathbb{R}^3$, $G^\lambda(x, y)$ has the explicit form $G^\lambda(x, y)=\exp(-\sqrt{\lambda}\,|x-y|)(4\pi\,|x-y|)^{-1}$.

We describe now the main features of the method by which the main result (Theorem 1 below) of this paper is obtained. First of all we need a "good" approximation for $G_m^\lambda$ which we obtain using an image-charge procedure, i.e., we approximate the Dirichlet boundary conditions on the obstacles with the "potential" due to point charges outside the integration region (in particular in the points $w_i$, $i=1,\ldots, m$). This approximation is proved then to become more and more precise in the limit in which the linear size of the obstacles goes to zero.

Specifically our aim is to write for the Green's function $G^\lambda_m(x, y; w^{(m)})$ an approximate expression of the type:

$$G^\lambda_m(x, y; \underline{w}^{(m)}) \sim G^\lambda(x, y) + \sum_{i=1}^{m} q^i_x(\underline{w}^{(m)})\, G^\lambda(w_i, y) \equiv H^\lambda_m(x, y; \underline{w}^{(m)})$$

with suitably chosen values for the charges $q^i$'s.

To fix the values of the $q^i$'s we will require that the average value of $H^\lambda_m$ on each $\partial B_j(\underline{w}^{(m)})$ be 0; apart from terms going rapidly to 0 this amounts to a requirement that

$$G^\lambda(x, w_j) + \sum_{\substack{i=1\\ i\neq j}}^{m} q^i_x G^\lambda(w_i, w_j) + \frac{q^j_x}{\alpha/m} = 0 \tag{1.1}$$

where $\alpha$ is the electrostatic capacity of $B$:

$$\alpha = \int_{\partial B} \left(\frac{\partial u}{\partial \hat{n}}\right)(z)\, dS(z)$$

where $\partial u/\partial \hat{n}$ denotes the component of the gradient of $u$ along the inner normal to $\partial B$ at $z$, $dS(z)$ indicates surface integration and $u$ is the unique solution of

$$(\Delta u)(x) = 0, \qquad x \in \mathbb{R}^3 \backslash \bar{B}$$

$$u(x) = 1, \qquad x \in \partial B$$

$$\lim_{|x| \to \infty} u(x) = 0$$

To clarify the intuitive idea under the choice of the condition (1.1) notice that by the above definition $\alpha/m$ is the total electrostatic charge corresponding to a potential equal to 1 on $\partial B_j(\underline{w}^{(m)})$. $(q^j/\alpha/m)$ is then the electrostatic $(\lambda = 0)$ potential corresponding to a total charge $q^j$ on the same surface. An easy scaling argument (see Section 3) shows that the fact that $\lambda \neq 0$ tends to become irrelevant in the limit. Analogously $G^\lambda(w_i, w_j)$ is the mean value of $G^\lambda(w_i, y)$ on $\partial B_j(\underline{w}^{(m)})$ only for $\lambda = 0$, by the mean value theorem, but the difference there between $G^\lambda(w_i, y)$ and $e^{-\sqrt{\lambda}|w_i - w_j|} G^0(w_i, y)$ tends rapidly to 0 as $m$ tends to infinity.

In the notation introduced previously (1.1) becomes

$$\mathbf{G}^\lambda_x + \mathbf{q}_x \left(\mathsf{G}^\lambda + \frac{m}{\alpha}\mathbb{1}\right) = 0 \tag{1.2}$$

where $\mathbb{1}$ is the unitary matrix in $\mathbb{R}^m$ and $\mathbf{q}_x = \{q^1_x, \ldots, q^m_x\}$.

To see that the relation (1.2) can be inverted to give $\mathbf{q}_x$ it is enough to show that $|||\mathsf{G}^\lambda/m|||^{(m)} \leqslant 1/\alpha$ for $m$ sufficiently large.

From

$$(|x| \sqrt{\lambda})^\beta e^{-\sqrt{\lambda}|x|} \leqslant 1 \qquad \forall \beta < 1$$

we have

$$\frac{e^{-\sqrt{\lambda}|w_i - w_j|}}{|w_i - w_j|} \leqslant \frac{\lambda^{-\beta/2}}{|w_i - w_j|^{1+\beta}}$$

Taking $\beta < 1/2$ and using the assumption $A_2$ we have for any configuration of points in $W^{(m)}$:

$$\left|\left|\left|\frac{\mathsf{G}^\lambda}{m}\right|\right|\right|^{(m)} \leqslant \frac{1}{m} \left\{ \sum_{\substack{i,j \\ i \neq j}} \frac{e^{-2\sqrt{\lambda}|w_i - w_j|}}{16\pi^2 |w_i - w_j|^2} \right\}^{1/2} \leqslant C\lambda^{-\beta/2} \tag{1.3}$$

For $\lambda$ sufficiently large and $\underline{w}^{(m)} \in W^{(m)}$ (1.2) is then invertible, uniformly in $x \in D(\underline{w}^{(m)})$, and yields

$$\mathbf{q}_x(\underline{w}^{(m)}) \equiv -\frac{\alpha}{m} \mathbf{G}_x^\lambda(\underline{w}^{(m)}) \left[ \frac{\alpha \mathsf{G}^\lambda}{m} (\underline{w}^{(m)}) + \mathbb{1} \right]^{-1} \tag{1.4}$$

Substituting into the definition of $H_m^\lambda$ we get

$$H_m^\lambda(x, y; \underline{w}^{(m)}) = G^\lambda(x, y) - \frac{\alpha}{m} \mathbf{G}_x^\lambda(\underline{w}^{(m)}) \left[ \frac{\alpha}{m} \mathsf{G}^\lambda(\underline{w}^{(m)}) + \mathbb{1} \right]^{-1} \mathbf{G}_y^\lambda(\underline{w}^{(m)}) \tag{1.5}$$

In analogy with the previous notation we define for any $f \in L^2(\mathbb{R}^3)$

$$\mathbf{q}_f(\underline{w}^{(m)}) = -\frac{\alpha}{m} \mathbf{G}_f^\lambda(\underline{w}^{(m)}) \left[ \frac{\alpha \mathsf{G}^\lambda}{m} (\underline{w}^{(m)}) + \mathbb{1} \right]^{-1}$$

Notice that

$$\begin{aligned} |||m^{-1/2}\mathbf{G}_f^\lambda|||^{(m)} &= \left[ \frac{1}{m} \sum_{i=1}^m (G^\lambda f)^2(w_i) \right]^{1/2} \\ &\leqslant \sup_{x \in \mathbb{R}^3} |G^\lambda f|(x) \leqslant C \, \|f\|_2 \end{aligned} \tag{1.6}$$

so that $|||\mathbf{q}_f|||^{(m)} \leqslant Cm^{-1/2} \, \|f\|_2$. In the last inequality of (1.6) we used a standard potential estimate (see, e.g., Ref. 15).

We are now in a position to state the main result of the paper and the essential content of the sections leading to its proof.

In Section 2 we consider the case when $B$ is a sphere of radius $\alpha/4\pi$ (so that its electrostatic capacity is just $\alpha$) and prove that $H_m^\lambda - G_m^\lambda$ converges in norm to zero faster than $m^{-1/2}$, as $m$ goes to infinity. This in fact will be a simple extension of a result in Ref. 12.

In Section 3 we prove that $H_m^\lambda$ converges in norm to $(-\Delta+\alpha V+\lambda)^{-1}$ and we characterize the fluctuations around the limit.

Section 4 is devoted to the discussion of some extensions and applications.

The main result we obtain in the paper is the following:

**Theorem 1.** Let $B$ be the sphere of radius $\alpha/4\pi$. With the notation introduced above we have the following:

(a) For any $\varepsilon>0$, $\delta>0$ $\exists m_0$ such that

$$\|m^\eta[G_m^\lambda(\underline{w}^{(m)})\,\chi_m(\underline{w}^{(m)})-A^\lambda]\|_2<\delta \tag{1.7}$$

for any $m>m_0$, any $\underline{w}^{(m)}$ belonging to a set of $\{V(x)\,dx\}^{\otimes m}$ measure $>1-\varepsilon$, and any $\eta<1/2$, $\lambda>0$. Here $A^\lambda\equiv(-\Delta+\alpha V+\lambda)^{-1}$, $\chi_m(\underline{w}^{(m)})$ is the characteristic function of $D(\underline{w}^{(m)})$ and $(G^\lambda\chi_m f)(x)$ is extended to all $\mathbb{R}^3$ setting its value equal to zero on $\mathbb{R}^3\backslash D(\underline{w}^{(m)})$.

(b) For each $g\in L^2(\mathbb{R}^3)$, the random field on $L^2(\mathbb{R}^3)$,

$$\xi_g^\lambda(f;\underline{w}^{(m)})\equiv m^{1/2}(f,[G_m^\lambda(\underline{w}^{(m)})\,\chi_m(\underline{w}^{(m)})-A^\lambda]\,g) \tag{1.8}$$

[where $(\cdot,\cdot)$ is the inner product in $L^2$ $(\mathbb{R}^3)$], converges in distribution to the Gaussian random field $\bar{\xi}_g^\lambda(f)$ of mean 0 and covariance:

$$\begin{aligned} &E(\bar{\xi}_g^\lambda(f)\,\bar{\xi}_g^\lambda(f'))\\ &\quad=\alpha^2[(A^\lambda f A^\lambda g, A^\lambda f' A^\lambda g)_{L_v^2}-(A^\lambda f, A^\lambda g)_{L_v^2}(A^\lambda f', A^\lambda g)_{L_v^2}] \end{aligned} \tag{1.9}$$

with $(\cdot,\cdot)_{L_v^2}=(\cdot, V\cdot)$.

(c) If in addition the region $\Omega$ is such that $\Delta^{-1}$ is compact (in particular if $\Omega$ is bounded) the following further result holds: Let $\gamma_n^{(m)}(\underline{w}^{(m)})$ be the $n$th eigenvalue of $-\Delta_m$ and $\gamma_n$ the $n$th eigenvalue of $-\Delta+\alpha V$ then, if $\gamma_n$ is not degenerate and $\varphi_n$ is the corresponding normalized eigenfunction, the random variable $m^{1/2}(\gamma_n-\gamma_n^{(m)})\equiv\Pi_n^{(m)}$ tends in distribution to a Gaussian random variable $\bar{\Pi}_n$ of mean 0 and

$$\{E(\bar{\Pi}_n^2)\}^{1/2}=\alpha[(\varphi_n^2,\varphi_n^2)_{L_v^2(\Omega)}-(\varphi_n,\varphi_n)_{L_v^2(\Omega)}^2]^{1/2} \tag{1.10}$$

## 2. EFFECTIVENESS OF THE APPROXIMATION OF THE GREEN'S FUNCTION

We want to prove the following:

**Theorem 2** (Ozawa[12]). $\|G_m^\lambda(\underline{w}^{(m)})-H_m^\lambda(\underline{w}^{(m)})\|_2^{(m)}<Cm^{-\beta}$ for any $\beta<2/3$ uniformly in $\underline{w}^{(m)}\in W^{(m)}$.

*Proof.* Take $f(y) \in C_0^\infty(D(\underline{w}^{(m)})$. $u_m(y) \equiv [(H_m^\lambda - G_m^\lambda) f](y)$ is a solution of $(-\Delta + \lambda) u_m = 0$ on $D(\underline{w}^{(m)})$, with $u_m(z) = (H_m^\lambda f)(z)$ for $z \in \bigcup_{i=1}^m \partial B_i(w^{(m)})$. We subtract from $u_m(z)$ the null term obtained integrating over $\mathbb{R}^3$ the left-hand side of (1.1) multiplied by $f$. We get

$$u_m(z) = [(G^\lambda f)(z) - (G^\lambda f)(w_i)] + \sum_{\substack{j=1\\ j\neq i}}^m q_f^j [G^\lambda(w_j, z) - G^\lambda(w_j, w_i)]$$

$$- q_f^i \frac{m}{\alpha} [1 - e^{-\sqrt{\lambda}(\alpha/4\pi m)}] \qquad \text{for} \quad z \in \partial B_i(\underline{w}^{(m)}) \tag{2.1}$$

where we wrote the explicit value of $G^\lambda(w_i, z)$ on $\partial B_i(\underline{w}^{(m)})$ in the last term.

By the assumptions $A_1$ and $A_2$ the first two terms in (2.1) are bounded by

$$M_m^i(f) \equiv \frac{C}{m} \left[ \sup_{y \in \mathbb{R}^3} |\nabla G^\lambda f| (y) + \sum_{\substack{j=1\\ j\neq i}}^m |q_f^j| \frac{1}{|w_i - w_j|^2} \right]$$

and the last one by $C\,|q_f^i|$.

The function

$$\bar{u}_m(y) = \frac{\alpha}{m} e^{\sqrt{\lambda}(\alpha/4\pi m)} \left[ \sum_{i=1}^m (M_m^i(f) + |q_f^i|)\, G^\lambda(w_i, y) \right]$$

is a solution of $(-\Delta + \lambda)\, \bar{u}_m(y) = 0$ on $D(\underline{w}^{(m)})$ such that $\bar{u}_m(z) \geqslant M_m^i + |q_f^i|$ for $z \in \partial B_i(\underline{w}^{(m)})$, for each $i$.

By the maximum principle $C\bar{u}_m(y)$ is then an upper bound for $u_m(y)$ on $D(\underline{w}^{(m)})$ and for any $g \in C_0^\infty(D(\underline{w}^{(m)})$ we have

$$|(g, (H_m^\lambda - G_m^\lambda) f)| \leqslant \frac{C}{m} \sup_{y \in \mathbb{R}^3} |\nabla G^\lambda f|\, (y) \sup_i G^\lambda\, |g|\, (w_i) + \frac{C}{m^2} |\mathbf{q}_f|\, \mathsf{Q}\mathbf{G}_{|g|}^\lambda$$

where $\mathsf{Q}$ is the matrix $\mathsf{Q}_{ij} = |w_i - w_j|^{-2}$ if $i \neq j$ and $\mathsf{Q}_{ii} = m$, $|\mathbf{q}_f| = \{|q_f^1|,..., |q_f^m|\}$. Applying the Schwartz inequality and making use of (1.6):

$$\begin{aligned} |(g, (H_m^\lambda - G_m^\lambda) f)| &\leqslant \frac{C}{m} \sup_{y \in \mathbb{R}^3} |\nabla G^\lambda f|\, (y) \sup_i G^\lambda |g|\, (w_i) \\ &\quad + \frac{C}{m^2} ||| \, |\mathbf{q}_f| \, |||^{(m)} \, |||\mathsf{Q}|||^{(m)} \, ||| \, \mathbf{G}_{|g|}^\lambda \, |||^{(m)} \\ &\leqslant \frac{C}{m^2} \|G^\lambda f\|_{C^1} [m + |||\mathsf{Q}|||^{(m)}] \sup_i G^\lambda |g|\, (w_i) \end{aligned} \tag{2.2}$$

where $\|G^\lambda f\|_{C^1} = \sup_{y \in \mathbb{R}^3} [|\nabla G^\lambda f|\, (y) + |G^\lambda f|\, (y)]$.

Notice that for any $0<\sigma<3/2$

$$G^{\lambda}|g|\ (w_i)\leqslant m^{\sigma}\int\frac{e^{-\sqrt{\lambda}|w_i-z|}}{4\pi\,|w_i-z|^{1-\sigma}}\,|g|\ (z)\,dz$$

for $g\in C_0^{\infty}(D(\underline{w}^{(m)}))$, so that by the potential estimates:

$$\sup_i G^{\lambda}|g|\ (w_i)\leqslant Cm^{\sigma}\ \|g\|_q^{(m)}\qquad\text{for}\quad \tfrac{3}{2}\geqslant q>\tfrac{3}{2}-\sigma$$

We have then

$$|(g,(H_m^{\lambda}-G_m^{\lambda})f)|\leqslant\frac{C}{m^{2-\sigma}}\,\|f\|_p^{(m)}\ \|g\|_q^{(m)}\ [m+|||\mathrm{Q}|||^{(m)}]\tag{2.3}$$

for any $p>3$ and $q>\frac{3}{2}-\sigma$ where we made use of the Sobolev inequality $\|G^{\lambda}f\|_{C^1}\leqslant\|f\|_p^{(m)}$ with $p>3$.[15]

By the definition of Q and the assumptions $A_1$, $A_2$ we get

$$\begin{aligned}|||\mathrm{Q}|||^{(m)}&\leqslant m+\Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{1}{|w_i-w_j|^4}\Bigg)^{1/2}\\ &\leqslant m+Cm^{(1-\nu)(1-\xi)/2}\Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{m}\frac{1}{|w_j-w_i|^{3-\xi}}\Bigg)^{1/2}\\ &\leqslant Cm^{(1-\nu)(1-\xi)/2+1}\end{aligned}\tag{2.4}$$

for any $\xi>0$ and $\nu<\frac{1}{3}$. Inserting the bound (2.4) in (2.3) we have

$$|(g,(H_m^{\lambda}-G_m^{\lambda})f)|\leqslant C\|f\|_p^{(m)}\ \|g\|_q^{(m)}\,m^{-1+\sigma+\sigma'}\tag{2.5}$$

for any $p>3$, $q>\frac{3}{2}-\sigma$ and $\sigma'>\frac{1}{3}$. (2.5) and the symmetry of $(H_m^{\lambda}-G_m^{\lambda})$ say that for any $\varepsilon>0$ there are $q<\frac{3}{2}$ and $q'=(1-1/q)^{-1}>3$ such that $(H_m^{\lambda}-G_m^{\lambda})$ is a bounded operator from $L_q(D(\underline{w}^{(m)}))$ to $L_q(D(\underline{w}^{(m)}))$ and from $L_{q'}(D(\underline{w}^{(m)})$ to $L_{q'}(D(\underline{w}^{(m)}))$ with norm less than $Cm^{-2/3+\varepsilon}$. Making use of the Riesz–Thorin lemma we finally conclude the proof of the theorem. ▮

***Remark.*** Using the fact that $G^{\lambda}|g|\ (w_i)$ is uniformly bounded in (2.2) the use of the Hilbert–Schmidt norm for (the nondiagonal part of) Q can be avoided. It is then possible to get a rate of convergence $m^{-\gamma}$ with $\gamma<1$.

Notice that (1.5) defines $H_m^{\lambda}$ as a bounded operator in all $L^2(\mathbb{R}^3)$. Moreover, it will be in $L^2(\mathbb{R}^3)$ that we will show, in the next section, the convergence of $H_m^\lambda$ to the limit operator $(-\Delta+\alpha V+\lambda)^{-1}$. From now on with $H_m^\lambda$ we will mean the operator (1.5) defined in all $L^2(\mathbb{R}^3)$. Its restriction to $L^2(D(\underline{w}^{(m)}))$, we considered until now, will be then $\chi_m H_m^\lambda \chi_m$ where $\chi_m(x; \underline{w}^{(m)})$ is the characteristic function of $D(\underline{w}^{(m)})$. An estimate of the difference between the two operators is given in the following:

**Proposition 1.** For $\underline{w}^{(m)} \in W^{(m)}$ we have

$$\|H_m^\lambda - \chi_m H_m^\lambda \chi_m\|_2 \leqslant C m^{-1/2} \tag{2.6}$$

Moreover for any $f \in L^2(\mathbb{R}^3)$,

$$\lim_{m\to\infty} \|m^{1/2}(H_m^\lambda - \chi_m H_m^\lambda \chi_m) f\|_2 = 0 \tag{2.7}$$

*Proof.* In fact,

$$\|H_m^\lambda - \chi_m H_m^\lambda \chi_m\|_2 \leqslant 2\,\|H_m^\lambda(1-\chi_m)\|_2$$

and for any $f$, $g$ belonging to $C_0^\infty(\mathbb{R}^3)$

$$\begin{aligned}|(f, H_m^\lambda(1-\chi_m)\,g)| &\leqslant \|\mathbf{q}_f\|^{(m)}\,\|\mathbf{G}^\lambda_{(1-\chi_m)g}\|^{(m)} + \|G^\lambda f\|_\infty\,\|(1-\chi_m)\,g\|_1\\ &\leqslant C\|f\|_2\left\{\frac{1}{m}\sum_{i=1}^m [G^\lambda(1-\chi_m)\,g]^2(w_i)\right\}^{1/2} + \frac{C}{m}\|f\|_2\,\|g\|_2\end{aligned} \tag{2.8}$$

Dividing the contributions from the various obstacles we have

$$\begin{aligned}|G^\lambda(1-\chi_m)\,g|\,(w_i) &\leqslant [G^\lambda(1-\chi_m)\,|g|](w_i)\\ &= \sum_{\substack{j=1\\ j\neq i}}^m (G^\lambda\tilde{\chi}_m^j\,|g|)(w_i) + (G^\lambda\tilde{\chi}_m^i\,|g|)(w_i)\end{aligned}$$

where $\tilde{\chi}_m^j$ is the characteristic function of $B_j(\underline{w}^{(m)})$. For $\underline{w}^{(m)} \in W^{(m)}$ we then get

$$\begin{aligned}|G^\lambda(1-\chi_m)\,g|\,(w_i) &\leqslant C\sum_{\substack{j=1\\ j\neq i}}^m \frac{1}{|w_i-w_j|}\,\|\tilde{\chi}_m^j\,|g|\,\|_1 + \sup_{y\in B_i(w^{(m)})} (G^\lambda\tilde{\chi}_m^i\,|g|)(y)\\ &\leqslant C\left(m^{-3/2}\sum_{\substack{j=1\\ j\neq i}}^m \frac{1}{|w_i-w_j|} + m^{-1/2}\right)\sup_i \|\tilde{\chi}_m^i\,g\|_2\end{aligned}$$

where the Schwartz inequality was applied to the first term and a potential estimates to the second in the last step. We have therefore

$$\begin{aligned}
&\left\{\frac{1}{m}\sum_{i=1}^{m}\left[G^{\lambda}(1-\chi_m)\,|g|\right]^2(w_i)\right\}^{1/2}\\
&\quad\leqslant C\left(m^{-4}\sum_{\substack{i,j,j'=1\\ j\neq i;j'\neq i}}^{m}\frac{1}{|w_i-w_j|}\,\frac{1}{|w_i-w_{j'}|}\right.\\
&\qquad\left.+\,m^{-3}\sum_{\substack{i,j\\ i\neq j}}^{m}\frac{1}{|w_i-w_j|}+m^{-1}\right)^{1/2}\sup_i\|\tilde{\chi}^i_m g\|_2\\
&\quad\leqslant Cm^{-1/2}\sup_i\|\tilde{\chi}^i_m g\|_2
\end{aligned}\tag{2.9}$$

where we made use of the assumption $A_2$ to bound the sums in parentheses.

Being $\|\tilde{\chi}^i_m g\|_2\leqslant\|g\|_2$ and $\lim_{m\to\infty}\sup_i\ \|\tilde{\chi}^i_m g\|_2\leqslant\lim_{m\to\infty}\|(1-\chi_m)\,g\|_2=0$ by the dominated convergence theorem, (2.6) and (2.7) of the proposition follow. ∎

## 3. CONVERGENCE AND FLUCTUATIONS

In this section we want to investigate the asymptotic properties of $H^{\lambda}_m$ when $m$ goes to infinity.

As before, denote by $V$ the operator of multiplication by $V(x)$.

From the definition of $H^{\lambda}_m$ we can see that the convergence of $H^{\lambda}_m$ to $A^{\lambda}\equiv(-\Delta+\alpha V+\lambda)^{-1}=(\alpha G^{\lambda}V+1)^{-1}\,G^{\lambda}$ corresponds formally to the convergence of the sequence of $m\times m$ matrices $[(\alpha\mathsf{G}^{\lambda}/m)(\underline{w}^{(m)})+\mathbb{1}]^{-1}$ on $l^2(\mathbb{R}^m)$ to the operator $(\alpha G^{\lambda}V+1)^{-1}$ on $L^2(\mathbb{R}^3)$. Using the resolvent expansion this is equivalent to convergence of $\{(\alpha/m)\,\mathsf{G}^{\lambda}(\underline{w}^{(m)})^n\}$ to $(\alpha G^{\lambda}V)^n$. Notice that for any continuous function $f$ and any $y\in\mathbb{R}^3$ the sequence $(\alpha/m)\sum_{j=1}^{m}G^{\lambda}(y,w_j)\,f(w_j)$ converges as $m\to\infty$ to $\alpha(G^{\lambda}Vf)(y)$, by the law of large numbers. We are then mainly left to prove that we have convergence of the $n$th power of the matrix $\alpha\mathsf{G}^{\lambda}/m$ to the $n$th power of the operator $\alpha G^{\lambda}V$. This would be an immediate consequence of the law of large numbers applied to $[(\alpha/m)\,\mathsf{G}^{\lambda}]^n$ if it were true that $E\{(\alpha/m)\,\mathsf{G}^{\lambda}]^n\}<\infty$, where $E$ indicates expectation with respect to $\{V(x)\,dx\}^{\otimes m}$. This is not the case for the presence in each matrix element $\{[(\alpha/m)\,\mathsf{G}^{\lambda}]^n\}_{ij}$ of factors $G^{\lambda}(w_i,w_{k_1})\cdots G^{\lambda}(w_{k_{n-1}},w_j)$ in which some point is repeated (one has for example the term $(\alpha/m)^n[G^{\lambda}(w_i,w_j)]^n$). The mean value of these terms is in general infinite.

We prove now a lemma that guarantees that the terms with repeated points are negligible for a set of configurations whose measure increases to 1 when $m$ goes to infinity. Associate to each term $G^\lambda(w_i, w_{k_1})$. $G^\lambda(w_{k_1}, w_{k_2}) \cdots G^\lambda(w_{k_{n-2}}, w_{k_{n-1}})\, G^\lambda(w_{k_{n-1}}, w_j)$ in $[(\mathsf{G}^\lambda)^n]_{ij}$ the corresponding path from $i$ to $j$ defined by the sequences $(w_i, w_{k_1} \cdots w_{k_{n-1}}, w_j)$ and define the matrices:

$$[(\mathsf{G}^\lambda)^n]_{ij} = [(\mathsf{G}^\lambda)^n_N]_{ij} + [(\mathsf{G}^\lambda)^n_I]_{ij}$$

where $[(\mathsf{G})^n_I]$ is the sum of the terms corresponding to paths from $i$ to $j$ that intersect themselves at least once and $[(\mathsf{G})^n_N]_{ij}$ the ones in which no loops are present. Accordingly the matrices $(\mathsf{G}^\lambda + \mathbb{1})^{-1}_I$ and $(\mathsf{G}^\lambda + \mathbb{1})^{-1}_N$ are then defined and

$$H^\lambda_{m_I}(x, y; \underline{w}^{(m)}) = -\frac{\alpha}{m}\, \mathbf{G}^\lambda_x(\underline{w}^{(m)}) \left[\frac{\alpha}{m}\, \mathsf{G}^\lambda(\underline{w}^{(m)}) + \mathbb{1}\right]^{-1}_I \mathbf{G}^\lambda_y(\underline{w}^{(m)})$$

$$H^\lambda_{m_N}(x, y; \underline{w}^{(m)}) = G^\lambda(x, y) - \frac{\alpha}{m}\, \mathbf{G}^\lambda_x(\underline{w}^{(m)}) \left[\frac{\alpha}{m}\, \mathsf{G}^\lambda(\underline{w}^{(m)}) + \mathbb{1}\right]^{-1}_N \mathbf{G}^\lambda_y(\underline{w}^{(m)})$$

We want to prove now the following:

**Lemma 1.** If $\lambda$ is taken large enough, for any $\varepsilon > 0$ one can find an $\bar{m}$ such that if $m \geqslant \bar{m}$ then $\|m^{1/2} H^\lambda_{m_I} f\|_2 < \varepsilon\, \|f\|_2$ for any $w^{(m)}$ belonging to $W^{(m)}$.

*Proof.* From the definition of $H^\lambda_{m_I}$

$$|(g, m^{1/2} H^\lambda_{m_I} f)| \leqslant \sum_{s=2}^{\infty} \frac{\alpha^{s+1}}{m^{s+1/2}}\, |\mathbf{G}^\lambda_g (\mathsf{G}^\lambda)^s_I\, \mathbf{G}^\lambda_f| \tag{3.1}$$

Let us consider the $s$th term $\mathbf{G}^\lambda_g (\mathsf{G}^\lambda)^s_I\, \mathbf{G}^\lambda_f$ and compute the total contribution due to terms relative to paths which have the first repeated point after $s_1$ steps, $0 \leqslant s_1 \leqslant s-2$, and a loop of $s_2$ steps, $2 \leqslant s_2 \leqslant (s - s_1)$

Explicitly,

$$A^m_{s_1 s_2}(g, f) \equiv \frac{\alpha^{s+1}}{m^{s+1/2}} \sum_{i,p,k,l,q,r,j=1}^{m} [\mathbf{G}^\lambda_g]_i [(\mathsf{G}^\lambda)^{s_1 - 1}_N]_{ip} [\mathsf{G}^\lambda]_{pk} [\mathsf{G}^\lambda]_{kl}$$

$$\times\ [\mathsf{G}^\lambda)^{s_2 - 2}]_{lq} [\mathsf{G}^\lambda]_{qk} [\mathsf{G}^\lambda]_{kr} [(\mathsf{G}^\lambda)^{s - s_1 - s_2 - 1}]_{rj} [\mathbf{G}^\lambda_f]_j \tag{3.2}$$

(For sake of notational simplicity we are not considering the case $s_1=0$ which can be nevertheless treated in exactly the same way.)

$$
\begin{aligned}
A^m_{s_1,s_2}(g,f) \leqslant\; & C\alpha^{s+1}\left[\frac{1}{m}\sum_{i=1}^{m}(G^\lambda g)^2(w_i)\right]^{1/2}\left(\left|\left|\left|\frac{\mathrm{G}^\lambda}{m}\right|\right|\right|^{(m)}\right)^{s-s_2-2} \\
& \times\left[\frac{1}{m}\sum_{j=1}^{m}(G^\lambda f)^2(w_j)\right]^{1/2} m^{-(s_2+3/2)}\left\{\sum_{p,k,r=1}^{m}[\mathrm{G}^\lambda]^2_{pk}[\mathrm{G}^\lambda]^2_{kr}\right\}^{1/2} \\
& \times\left\{\sum_{k=1}^{m}\left[\sum_{l,q=1}^{m}[\mathrm{G}^\lambda]_{kl}[(\mathrm{G}^\lambda)^{s_2-2}]_{l,q}[\mathrm{G}^\lambda]_{qk}\right]^2\right\}^{1/2} \\
\leqslant\; & C\alpha^{s+1}\,\|f\|^2\,\|g\|_2\left(\left|\left|\left|\frac{\mathrm{G}^\lambda}{m}\right|\right|\right|^{(m)}\right)^{s-4} m^{-7/2}\sum_{p,k,r}[\mathrm{G}^\lambda]^2_{pk}[\mathrm{G}^\lambda]^2_{kr}
\end{aligned}
\tag{3.3}
$$

where again we used $\sup|G^\lambda g|\leqslant C\,\|g\|_2$.

In the last multiple sum we consider separately the contributions corresponding to $p=r$ and to $p\neq r$:

$$
\begin{aligned}
\eta(w^{(m)}) \equiv\; & m^{-7/2}\left\{\sum_{\substack{p,k,r\\ p\neq k\\ p\neq r\\ k\neq r}}[G^\lambda(w_p,w_k)]^2[G^\lambda(w_k,w_r)]^2+\sum_{\substack{p,k\\ p\neq k}}[G^\lambda(w_p,w_k)]^4\right\} \\
\leqslant\; & m^{-1/2}\left(m^{-3}\sum_{\substack{p,k,r\\ p\neq k\\ p\neq r\\ k\neq r}}\frac{1}{|w_p-w_k|^2}\frac{1}{|w_k-w_r|^2}+m^{-3}\sum_{\substack{k,p\\ k\neq p}}\frac{1}{|w_p-w_k|^4}\right)
\end{aligned}
$$

By (2.4) and by explicit computation of the mean value of the first sum we immediately get that the term in parantheses is finite almost everywhere in $W^{(m)}$ so that $\lim_{m\to\infty}\eta(\underline{w}^{(m)})=0$ for almost any sequence $\underline{w}^{(m)}$, $\underline{w}^{(m)}\in W^{(m)}$.

Using (1.3):

$$
A^m_{s_1s_2}(g,f)\leqslant\|f\|_2\,\|g\|_2\,(C\alpha\lambda^{-\beta|2})^{s-4}\,\eta(\underline{w}^{(m)})
$$

Being the estimate independent of $s_1$ and $s_2$ we have only to count the possible positions and lengths of the loop. They are $\sum_{s_2=2}^{s}(s-s_2-1)\leqslant s^2$ so that we finally get

$$
|(g,m^{1/2}H^\lambda_{m_l}f)|\leqslant C\,\|f\|_2\,\|g\|_2\,\eta(\underline{w}^{(m)})\sum_{s=2}^{\infty}(C\lambda^{-\beta/2})^s s^2 \tag{3.4}
$$

which converges to 0 as $m$ goes to infinity if $\lambda$ is large enough to make the series on the right-hand side of (3.4) converge. ▮

By Lemma 1 we can substitute $H_m^\lambda$ by $H_{m_N}^\lambda$ if we want to establish convergence in probability. Moreover, $H_{m_N}^\lambda(x, y; w^{(m)})$ is a random variable in $(\mathbb{R}^3)^m$ which is invariant under permutations of the $w_i$'s and has finite first and second moment. A law of large number and a central limit theorem should then hold for it.

*Proof of Theorem* 1. We first prove part (a). Let the random variable $\xi_g^\lambda(f; \underline{w}^{(m)})$ be defined as in the statement of Theorem 1 in the introduction. The result we have established so far can be restated as follows: almost everywhere in $W^{(m)}$ one has

$$\begin{aligned}\lim_{m\to\infty} \xi_g^\lambda(f; \underline{w}^{(m)}) &= \lim_{m\to\infty} m^{1/2}(f, H_{m_N}^\lambda(\underline{w}^{(m)})\, g - A^\lambda g)\\ &= \lim_{m\to\infty} \sum_{s=1}^{m} (-\alpha)^s \{m^{-s+1/2}\mathbf{G}_f^\lambda(\underline{w}^{(m)})[\mathsf{G}^\lambda(\underline{w}^{(m)})]_N^{s-1}\\ &\quad \times \mathbf{G}_g^\lambda(\underline{w}^{(m)}) - m^{1/2}(f, G^\lambda(VG^\lambda)^s g)\}\end{aligned} \tag{3.5}$$

[Notice that we are allowed to truncate the sum at the $m$th term by the lemma and by the norm convergence of the Neumann series for $(-\Delta+\lambda+\alpha V)^{-1}$.]

Denote by $\Theta_g^\lambda(f; \underline{w}^{(m)})$ the random variable given by the sum on the right-hand side of (3.5). Then $\lim_{m\to\infty} E(\Theta_g^\lambda(f; \underline{w}^{(m)}))=0$ and for any $f, f' \in L^2(\mathbb{R}^3)$ we can compute explicitly:

$$\begin{aligned}&E(\Theta_g^\lambda(f)\, \Theta_g^\lambda(f'))\\ &= \sum_{s,s'=1}^{m} \{(-\alpha)^{s+s'} E[m^{-s-s'+1}\mathbf{G}_f^\lambda(\mathsf{G}^\lambda)_N^{s-1}\, \mathbf{G}_g^\lambda \mathbf{G}_{f'}^\lambda(\mathsf{G})_N^{s'-1}\mathbf{G}_g^\lambda]\\ &+\left[m - \frac{m!\, m^{-s+1}}{(m-s)!} - \frac{m!\, m^{-s'+1}}{(m-s')!}\right](f, G^\lambda(VG^\lambda)^s g)(f', G^\lambda(VG^\lambda)^{s'} g)\Big\}\end{aligned} \tag{3.6}$$

Again we will distinguish the cases: (1) pairs of paths of $(\mathsf{G}^\lambda)_N^{s-1}$ and $(\mathsf{G}^\lambda)_N^{s'-1}$ which have no intersections; (2) pairs of paths that intersect once; (3) pairs of paths which have more than one intersection.

Considering only paths of type (1), (3.6) becomes

$$\begin{aligned}&\sum_{s,s'=1}^{m} (-\alpha)^{s+s'}\left[m - \frac{m!\, m^{-s+1}}{(m-s)!} - \frac{m!\, m^{-s'+1}}{(m-s')!} + \frac{m!\, m^{-s-s'+1}}{(m-s-s')!}\right]\\ &\quad\times (f, G^\lambda(VG^\lambda)^s g)(f', G^\lambda(VG^\lambda)^{s'} g)\end{aligned} \tag{3.7}$$

where $[(m-s-s')!]^{-1}$ is intended to be 0 for $s+s'>m$.

Let $\zeta(m, s, s')$ denote the combinatorial term in (3.7). One easily verifies that $|\zeta(m, s, s') + ss'| \leqslant C(s^4 s'^4/m)$, so that

$$(3.7) = -\left(f, \left[\sum_{s=1}^{m} (-\alpha)^s sG^\lambda(VG^\lambda)^s\right] g\right)\left(f', \left[\sum_{s'=1}^{m} (-\alpha)^{s'} s'G^\lambda(VG^\lambda)^s\right] g\right)$$
$$+ O\left(\frac{1}{m}\right) \sum_{s,s'=1}^{m} (\alpha)^{s+s'} s^4 s'^4 \, |(f, G^\lambda(VG^\lambda)^s g)| \, |(f', G^\lambda(VG^\lambda)^{s'} g)| \tag{3.8}$$

where the last series converges uniformly in $\|f\|_2 = \|f'\|_2 = 1$ for $\lambda$ sufficiently large.

The contribution of terms of type (2) is

$$\sum_{s,s'=1}^{m} (-\alpha)^{s+s'} \frac{m!\, m^{-s-s'+1}}{(m-s-s'+1)!}$$
$$\times \sum_{n=0}^{s-1} \sum_{p=0}^{s'-1} E\left\{\sum_{i=1}^{m} [G^\lambda(VG^\lambda)^n f](w_i)[G^\lambda(VG^\lambda)^{s-n-1} g](w_i)\right.$$
$$\left. \times [G^\lambda(VG^\lambda)^p f'](w_i)[G^\lambda(VG^\lambda)^{s'-p-1} g](w_i)\right\}$$
$$= \alpha^2 \left(\left\{G^\lambda\left[\sum_{n=0}^{m-1} (-\alpha)^n (VG^\lambda)^n f\right]\right\}\left\{G^\lambda\left[\sum_{p=0}^{m-1} (-\alpha)^p (VG^\lambda)^p g\right]\right\}\right.$$
$$\left\{G^\lambda\left[\sum_{n'=0}^{m-n-1} (-\alpha)^{n'} (VG^\lambda)^{n'} f'\right]\right\}$$
$$\left. \times \left\{G^\lambda\left[\sum_{p'=0}^{m-p-1} (-\alpha)^{p'} (VG^\lambda)^{p'} g\right]\right\}\right)_{L_v^2} + O\left(\frac{1}{m}\right) \tag{3.9}$$

where $(\cdot, \cdot)_{L_v^2} = (\cdot, V\cdot)$.

In the same way one verifies that the contribution of terms of type (3) is of order $m^{-1}$.

Adding (3.8) to (3.9) and taking into account that

$$\sum_{s=1}^{\infty} (-\alpha)^s sG^\lambda(VG^\lambda)^s = G^\lambda \alpha VG^\lambda \frac{1}{(1+\alpha VG^\lambda)^2}$$
$$= \alpha G^\lambda(1+\alpha VG^\lambda)^{-1} VG^\lambda(1+\alpha VG^\lambda)^{-1} = \alpha A^\lambda VA^\lambda \tag{3.10}$$

we get

$$\lim_{m\to\infty} E[\Theta_g^\lambda(f)\, \Theta_g(f')] = \text{r.h.s. of } (1.9).$$

Notice that this result implies in particular that

$$m^{\gamma}\,\|H^{\lambda}_{m_N}(\underline{w}^{(m)})-A^{\lambda}\|_2 \xrightarrow[m\to\infty]{} 0$$

in probability for any $\gamma<1/2$ and $\lambda$ sufficiently large. This result together with Theorem 2, (2.6) and Lemma 1 concludes the proof of the norm resolvent convergence to the limit operator $A^{\lambda}$ stated in the theorem.

The nature of the cancellations in the computation of $E[\Theta^{\lambda}_g(f)\,\Theta^{\lambda}_g(f')]$ suggests that the random field $\Theta$ has, in the limit, the same covariance matrix of the random field:

$$\xi^{\lambda}_g(f;\underline{w}^{(m)})\equiv m^{-1/2}\sum_{i=1}^{m}K^{\lambda}_g(f;w_i)-m^{1/2}E[K^{\lambda}_g(f;w_j)] \qquad (3.11)$$

where

$$K^{\lambda}_g(f;w_i)=\sum_{s=1}^{\infty}(-\alpha)^s\sum_{n=0}^{s-1}[G^{\lambda}(VG^{\lambda})^n f](w_i)[G^{\lambda}(VG^{\lambda})^{s-n-1}g](w_i)$$

and [see (3.10)]

$$E[K^{\lambda}_g(f;w_j)]=\alpha(A^{\lambda}f,A^{\lambda}g)_{L^2_V}$$

Notice that the sequence $\{\xi^{\lambda}_g(f;w^{(m)})\}$ converges to a Gaussian random variable since its elements are normalized sum of independent identically distributed random variables. Moreover,

$$E[\xi^{\lambda}_g(f)\,\xi^{\lambda}_g(f')]=E[K^{\lambda}_g(f)\,K^{\lambda}_g(f')]-E[K^{\lambda}_g(f)]\,E[K^{\lambda}_g(f')]$$

where the first term is identical with (3.9) up to a term of order $m^{-1}$.

To conclude the proof of part (b) it is then enough to show that $E\{[\Theta^{\lambda}_g(f)-\xi^{\lambda}_g(f)]^2\}$ converges to 0 when $m$ goes to infinity. Since the variances have the same limit, this amounts to prove that

$$\lim_{m\to\infty}E[\Theta^{\lambda}_g(f;\underline{w}^{(m)})\,\xi^{\lambda}_g(f;\underline{w}^{(m)})]=\lim_{m\to\infty}E\{[\xi^{\lambda}_g(f;\underline{w}^{(m)})]^2\} \quad (3.12)$$

By a straightforward computation entirely identical to the one made before for $E[\Theta^{\lambda}_g(f)\,\Theta^{\lambda}_g(f')]$ one shows that (3.12) holds. This concludes the proof of part (b).

To prove part (c) of the theorem, let us consider now the case when $\Omega$ is such that $\Delta^{-1}$ is compact. In particular $\Omega$ could be a bounded region with smooth boundaries. The function $V(x)$ has obviously support contained in $\Omega$. As we mentioned before the results we have proved so far for the case $\Omega = \mathbb{R}^3$ are still valid with slight modifications of the proofs, since they depend only on a bound on the (singular) behavior of $G^\lambda(x, y)$ when $|x-y| \to 0$.

With our assumptions on $V$ and on $\Omega$, $G_m^\lambda$ and $A^\lambda$ are compact operators for any $\lambda \geqslant 0$ and the following properties of their spectrum are well known:

(i) $-\Delta_m$ and $-\Delta$ have only discrete spectrum in $\mathbb{R}^+$. Each eigenvalue has finite multiplicity.

Denote by $\gamma_n^{(m)}(\underline{w}^{(m)})$ the $n$th eigenvalue (repeated according to the multiplicity) of $-\Delta_m$ and by $\gamma_n$ the $n$th eigenvalue of $-\Delta + \alpha V$. Let $P_n$ be the orthogonal projection onto the eigenspace relative to $\gamma_n$.

(ii) The norm convergence of $(-\Delta_m + \lambda)^{-1}$ to $(-\Delta + \alpha V + \lambda)^{-1}$ implies in particular that for (almost all) sequences $\{\underline{w}^{(m)}\}$, $\underline{w}^{(m)} \in W^{(m)}$, $\gamma_n^{(m)}(\underline{w}^{(m)}) \xrightarrow[m \to \infty]{} \gamma_n$. Moreover if $\mu_n$ is the multiplicity of $\gamma_n$, one can find a neighborhood $\Xi_n$ of $\gamma_n$ such that, for $m$ sufficiently large, $\Xi_n$ contains $r \leqslant \mu_n$ eigenvalues of $-\Delta_m$ of total multiplicity equal to $\mu_n$. The projection on the direct sum of their eigenspaces converges in norm to $P_n$ as $m$ goes to infinity, for almost all sequences $\{\underline{w}^{(m)}\}$, $\underline{w}^{(m)} \in W^{(m)}$.

We will consider here only the case when $\gamma_n$ has multiplicity 1; the case when $\gamma_n$ has multiplicity greater than 1 is not a straightforward extension of the one we will consider.

Let $\varphi_n$ be the (real) normalized eigenfunction corresponding to $\gamma_n$ and $\varphi_n^{(m)}$ the ones corresponding to $\gamma_n^{(m)}(\underline{w}^{(m)})$. One has

$$(\varphi_n, G_m^\lambda \chi_m \varphi_n^{(m)}) = \frac{1}{\gamma_n^{(m)} + \lambda} (\varphi_n, \varphi_n^{(m)}) \tag{3.13}$$

and

$$(\varphi_n, A^\lambda \varphi_n^{(m)}) = \frac{1}{\gamma_n + \lambda} (\varphi_n, \varphi_n^{(m)}) \tag{3.14}$$

Subtracting (3.14) from (3.13),

$$(\gamma_n - \gamma_n^{(m)})(\varphi_n, \varphi_n^{(m)}) = (\gamma_n + \lambda)(\gamma_n^{(m)} + \lambda)(\varphi_n, (G_m^\lambda \chi_m - A^\lambda)\, \varphi_n^{(m)})$$

By the norm convergence, $|(\varphi_n, \varphi_n^{(m)})| > 0$ for $m$ sufficiently large; we have then

$$\Pi_n^{(m)} \equiv m^{1/2}(\gamma_n - \gamma_n^{(m)}) = \frac{(\gamma_n + \lambda)(\gamma_n^{(m)} + \lambda)}{(\varphi_n, \varphi_n^{(m)})}\, \xi^\lambda_{\varphi_n^{(m)}}(\varphi_n)$$

On every $\underline{w}^{(m)} \in W^{(m)}$ the random variable $\xi^{\lambda}_{\varphi_n^{(m)}}(\varphi_n)$ differs from $\xi^{\lambda}_{\varphi_n}(\varphi_n)$ by a term which tends to 0 when $m \to \infty$.

Since $(\gamma_n + \lambda)(\gamma_n^{(m)} + \lambda)/(\varphi_n, \varphi_n^{(m)}) \xrightarrow[m \to \infty]{} (\gamma_n + \lambda)^2$, the convergence in distribution of the field $\xi_g(f)$ implies then the convergence in distribution of the random variables $\Pi_n^{(m)}$ to the Gaussian random variable $\bar{\Pi}_n$ of mean 0 and variance:

$$E(\bar{\Pi}_n^2) = \alpha^2[(\varphi_n^2, V\varphi_n^2) - (\varphi_n, V\varphi_n)^2] \quad \blacksquare$$

## 4. EXTENSIONS AND APPLICATIONS

We want to discuss briefly in this section some simple extensions of the results stated in Theorem 1. In particular (A) the two-dimensional case, (B) the case in which $B$ is not a sphere, and (C) release of some assumption on $V$.

(A) Owing to the logarithmic singularity of the Green's function in the two-dimensional case the limit of constant capacity is the one in which the radius of the disk is taken to be $e^{-m/2\pi\alpha}$. If this is the case, the definition (1.1) of the "charges" $\mathbf{q}_x$ turns out to be exactly the same. The only further feature of the proof of Theorem 1 was the possibility to find a set $W^{(m)}$ of measure going to 1 in the limit $m \to \infty$ such that (i) for any configuration of $W^{(m)}$ no two spheres intersect; (ii) the Hilbert–Schmidt norm of the matrix $\mathsf{G}_m^{\lambda}(\underline{w}^{(m)})$ is finite for $\underline{w}^{(m)} \in W^{(m)}$.

In the two-dimensional case it is then enough to require

$$A_1': \quad \min_{i,j} |w_i - w_j| > m^{-\rho} \qquad \text{for some} \quad \rho > 0$$

$$A_2': \quad \frac{1}{m^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \log^2 |w_i - w_j| < C < \infty$$

One checks immediately that for any continuous distribution $V(x)$, $A_1'$ and $A_2'$ hold in a set of measure increasing rapidly to 1 as $m$ goes to infinity.

The proof given in Section 3 can then be repeated without change.

It should be noticed that what prevents the extension of the proof to dimensions greater than 3 is the requirement that the Hilbert–Schmidt norm of matrices $\mathsf{G}_m^{\lambda}(\underline{w}^{(m)})$ be uniformly bounded.

(B) As in the Introduction let $B$ now a simply connected open neighborhood of the origin with $C^2$ boundary. Along the line of Lemma 1 in Ref. 10 we want to prove the following:

**Theorem 2bis).** (a) For any sequences $\{\underline{w}^{(m)}\}$, $\underline{w}^{(m)} \in W^{(m)}$,

$$\lim_{m \to \infty} m^{\beta} \, \|G_m^{\lambda}(\underline{w}^{(m)}) - H_m^{\lambda}(\underline{w}^{(m)})\|_2^{(m)} = 0$$

for any $\beta < 1/2$. Moreover, for any $g, f$ belonging to $H^1(\Omega)$

$$m^{1/2}(g, [G_m^\lambda(\underline{w}^{(m)}) - H_m^\lambda(\underline{w}^{(m)})]\, f) \leqslant \mathbb{C}(\underline{w}^{(m)})\, \|g\|_{H^1}\, \|f\|_{H^1}$$

where $\lim_{m \to \infty} \mathbb{C}(\underline{w}^{(m)}) = 0$. Here $H^1$ is the standard Sobolev space (form domain of $-\Delta$), and $\|\cdot\|_{H^1}$ denotes the corresponding norm.

(b) For each $g \in H^1$, the random field $\xi_g^\lambda(f; \underline{w}^{(m)})$ defined as in (1.8) for $f \in H^1$ converges in distribution to the Gaussian random field $\bar{\xi}_g^\lambda(f)$ of mean zero and covariance given by (1.9).

(c) The statement (c) in Theorem 1 holds also in the present case.

*Proof.* We notice that all the proofs of Section 2 are algebraic and do not refer to a special shape of the obstacles. Only in the proof of Theorem 2 [formula (2.1)] did we make explicit use of the value of $G^\lambda(w_i, y)$ for $y$ on $\partial B_i(\underline{w}^{(m)})$. Therefore (b) and (c) of Theorem 2bis are proved if we prove (a). For part (c) remark that if $\varphi_n$ is the eigenvector of $-\Delta + \alpha V$ associated to an eigenvalue $\gamma_n$ of multiplicity 1, and if $V \geqslant 0$, then $\varphi_n \in H^1$. We prove now part (a) of Theorem 2bis.

Consider first the function $v_i^{(m)}(y; \lambda)$ which satisfies

$$\begin{cases} [(-\Delta + \lambda)\, v_i^{(m)}](y; \lambda) = 0, & y \in \mathbb{R}^3 \backslash \bar{B}_i(\underline{w}^{(m)}) \\ v_i^{(m)}(z; \lambda) = \dfrac{m}{\alpha}, & z \in \partial B_i(\underline{w}^{(m)}) \\ \lim\limits_{|y| \to \infty} v_i^{(m)}(y; \lambda) = 0 \end{cases} \tag{4.1}$$

We have $v_i^{(m)}(y; \lambda) = (m/\alpha)\, \bar{v}(m(y - w_i); \lambda/m^2)$, where $\bar{v}(y, \mu)$ satisfies

$$\begin{cases} [(-\Delta + \mu)\, \bar{v}](y, \mu) = 0, & y \in \mathbb{R}^3 \backslash \bar{B} \\ \bar{v}(z, \mu) = 1, & z \in \partial B \\ \lim\limits_{|y| \to \infty} \bar{v}(y; \mu) = 0 \end{cases}$$

Green's second identity gives us [see the notation introduced in (1.1)]:

$$\begin{aligned} \frac{m}{\alpha}\, \bar{v}\left(my, \frac{\lambda}{m^2}\right) = {} & \frac{m}{\alpha} \int_{\partial B} G^{\lambda/m^2}(my, z) \left(\frac{\partial \bar{v}}{\partial \hat{n}}\right)\left(z; \frac{\lambda}{m^2}\right) dS(z) \\ & - \frac{m}{\alpha} \int_{\partial B} \left(\frac{\partial G^{\lambda/m^2}}{\partial \hat{n}}\right)(my, z)\, dS(z) \end{aligned}$$

$$
\begin{aligned}
&= G^{\lambda}(y,0) + \frac{1}{\alpha}\int_{\partial B}\left[G^{\lambda}\left(y,\frac{z}{m}\right) - G^{\lambda}(y,0)\right]\left(\frac{\partial \bar{v}}{\partial \hat{n}}\right)\left(z;\frac{\lambda}{m^2}\right) dS(z) \\
&\quad + \frac{G^{\lambda}(y,0)}{\alpha}\int_{\partial B}\left[\left(\frac{\partial \bar{v}}{\partial \hat{n}}\right)\left(z;\frac{\lambda}{m^2}\right) - \left(\frac{\partial \bar{v}}{\partial \hat{n}}\right)(z;0)\right] dS(z) \\
&\quad - \frac{m}{\alpha}\int_{\partial B}\left[\left(\frac{\partial G^{\lambda/m^2}}{\partial \hat{n}}\right)(my,z) - e^{-\sqrt{\lambda}|y|}\left(\frac{\partial G^{0}}{\partial \hat{n}}\right)(my,z)\right] dS(z) \\
&= G^{\lambda}(y,0) + \frac{1}{\alpha}\int_{\partial B}\left[G^{\lambda}\left(y,\frac{z}{m}\right) - G^{\lambda}(y,0)\right]\left(\frac{\partial \bar{v}}{\partial \hat{n}}\right)\left(z;\frac{\lambda}{m^2}\right) dS(z) \\
&\quad + \frac{G^{\lambda}(y,0)}{\alpha}\frac{\lambda}{m^2}\int_{\mathbb{R}^3\setminus\bar{B}} |\bar{v}|^2\left(x;\frac{\lambda}{m^2}\right) dx \\
&\quad + \frac{G^{\lambda}(y,0)}{\alpha}\int_{\mathbb{R}^3\setminus\bar{B}} \nabla\bar{v}\left(x;\frac{\lambda}{m^2}\right)\cdot\nabla\left(\bar{v}\left(x;\frac{\lambda}{m^2}\right) - \bar{v}(x,0)\right) dx \\
&\quad + \frac{\sqrt{\lambda}}{m\alpha}\int_{\partial B} G^{\lambda}\left(y,\frac{z}{m}\right)\frac{(y-z/m)}{|y-z/m|}\cdot n(z)\, dS(z) \\
&\quad - \frac{1}{m\alpha}\int_{\partial B}\left(e^{-\sqrt{\lambda}|y-z/m|} - e^{-\sqrt{\lambda}|y|}\right)\frac{\partial G^0}{\partial \hat{n}}\left(y,\frac{z}{m}\right) dS(z) \qquad (4.2)
\end{aligned}
$$

where we used repeatedly the Green's identities, the definition of capacity, and the fact that $\int_{\partial B} (\partial G^0/\partial \hat{n})(y,z)\, dS(z) = 0$ for $y \in \mathbb{R}^3\setminus\bar{B}$.

From (4.2) we get immediately the following results: (i) for any $y \neq 0$

$$
\left|\frac{m}{\alpha}\bar{v}\left(my;\frac{\lambda}{m^2}\right) - G^{\lambda}(0,y)\right| \leqslant \frac{C}{m}\left(\frac{1}{|y|} + \frac{1}{|y|^2}\right) \qquad (4.3)
$$

(ii) for any $g \in C_0^{\infty}(D(\underline{w}^{(m)}))$

$$
\left|\int\left[\frac{m}{\alpha}\bar{v}\left(my;\frac{\lambda}{m^2}\right) - G^{\lambda}(0,y)\right] g(y)\, dy\right| \leqslant \frac{C}{m}\,\|G^{\lambda}g\|_{C^1} \qquad (4.4)
$$

and (iii)

$$
\left|\int\left[\frac{m}{\alpha}\bar{v}\left(my;\frac{\lambda}{m^2}\right) - G^{\lambda}(0,y)\right] g(y)\, dy\right| \leqslant C \sup_{z\in\partial B}\left|(G^{\lambda}g)\left(\frac{z}{m}\right) - (G^{\lambda}g)(0)\right| \qquad (4.5)
$$

The last two results allow us to conclude the proof of Theorem 2bis) along the line of Theorem 2. In fact, for any $f \in C_0^\infty(D(\underline{w}^{(m)}))$, one has [compare (2.1)]

$$[(H_m^\lambda - G_m^\lambda) f(z)] \equiv u_m(z) = [(G^\lambda f)(z) - (G^\lambda f)(w_i)] + \sum_{\substack{j=1 \\ j \neq i}}^{m} q_f^j [G^\lambda(w_j, z) - G^\lambda(w_j, w_i)] + q_f^i \left[ G^\lambda(w_i, z) - \frac{m}{\alpha} \right] \qquad \forall z \in \partial B_i(w^{(m)}) \quad (4.6)$$

Let us consider the function

$$u_m^{(3)}(y) \equiv \sum_{i=1}^{m} [G^\lambda(w_i, y) - v_i^{(m)}(y; \lambda)]\, q_f^i$$

with the $v_i^{(m)}$'s defined by (4.1). On $\partial B_i(\underline{w}^{(m)})$

$$u_m^{(3)}(z) = q_f^i \left[ G^\lambda(w_i, z) - \frac{m}{\alpha} \right] + \sum_{\substack{j=1 \\ j \neq i}}^{m} [G^\lambda(w_j, z) - v_j^{(m)}(z; \lambda)]\, q_f^j, \qquad z \in \partial B_i(\underline{w}^{(m)})$$

But we have

$$\left| \sum_{\substack{j=1 \\ j \neq i}}^{m} [G^\lambda(w_j, z) - v_j^{(m)}(z; \lambda)]\, |q_f^j| \right| \leqslant \frac{C}{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} |q_f^j| \left( \frac{1}{|w_j - w_i|} + \frac{1}{|w_j - w_i|^2} \right)$$

which in the limit is smaller than the second term in (4.6).

We are then allowed to write

$$|(H_m^\lambda - G_m^\lambda) f|\,(y) \leqslant C(u_m^{(1)}(y) + u_m^{(2)}(y) + |u_m^{(3)}(y)|)$$

where $u_m^{(i)}(y)$, $i = 1, 2$ are both solutions of $[(-\Delta + \lambda)\, u_m^{(i)}](y) = 0$ on $D(\underline{w}^{(m)})$, with boundary values on $\partial B_i(\underline{w}^{(m)})$:

$$u_m^{(1)}(z) = \sup_{z \in \partial B_i(\underline{w}^{(m)})} |(G^\lambda f)(z) - (G^\lambda f)(w_i)|$$

$$u_m^{(2)}(z) = \sum_{\substack{j=1 \\ j \neq i}}^{m} |q_f^j| \frac{1}{|w_i - w_j|^2}, \qquad z \in \partial B_i(\underline{w}^{(m)})$$

Finally by (2.2), (2.4) of Theorem 2 and by (4.5),

$$m^{1/2}\, |(g, (H_m^\lambda - G_m^\lambda) f)| \leqslant C\, \|G^\lambda f\|_{C^1}\, \|G^\lambda g\|_{C^1}\, m^{-1/2 + \sigma'}$$

In particular $f, g \in H^1$ implies $f, g \in L^6$ so that $G^\lambda f$, $G^\lambda g \in W^{2,6}$, which is embedded in $C^1$ (see, e.g., Ref. 15).

On the other hand, if in Theorem 2 we do not make use of the symmetry we still obtain the result with a rate of convergence less then 1/2 corresponding to the maximal degree of Hölder continuity of $G^\lambda f$ for $f \in L^2$. For the same reason a rate of convergence less than 1/2 is obtained by (4.5) concluding the proof of Theorem 2bis). ∎

(C) We would like now to make some remarks concerning the regularity conditions on $V(x)$. The assumptions on $V$ are only requested to guarantee the validity of $A_1$ and $A_2$ on a set of configurations of measure increasing to 1 when $m$ goes to infinity. It is easily verified that the law of large numbers guarantees the validity of $A_2$ even for more general distribution functions $V$ (for example $V \in L_1 \cap L_2$). Unfortunately, the requirement that the obstacles have empty intersection is not fulfilled if $V$ is too singular. In particular $V$ must belong to $L_p$, $p > 3$, if one wants to avoid the possibility of "strong packing" of the obstacles.

It is to be noticed that this difficulty concerns the method we used and we do not exclude that the result could be true as well for any $V \in L_1 \cap L_2$. The image-charges technique we describe in this paper tends in fact to lose its effectiveness when the obstacles are not well distinct (with large probability).

We want finally to make a comment about the relation between our discussion of randomly placed small scatterers and some recent work on point interactions (see, e.g., Ref. 16 and references therein). The latter work concerns attractive $\delta$ potentials, used in particular for the study of the low-energy behavior of quantum mechanical scattering quantities. The necessity of the attractiveness is due to the fact that discrete sets of points are considered and a perturbation of the Laplacian in $\mathbb{R}^3$ with support in a discrete set of points is known to be nontrivial only in the attractive case. On the other hand, by the method described above it is possible to construct a (large) finite number of hard core potentials by starting, as a zero approximation, from an infinite number of scatterers and expanding then in the inverse powers of the number of scatterers.

We conclude with some short discussion about possible further applications of the above results.

In Refs. 2 and 7–9 was already pointed out the possibility to use the convergence result in problems of quantum or light scattering by randomly placed hard cores (low-energy scattering by glasses or scattering of neutrons by disordered cristals could be examples). A problem of multiple scattering is firstly reduced to scattering by an effective potential. The fluctuations give then a first-order term in an expansion around the $m = \infty$

limit. Both terms contain as a physical parameter the linear size per unit volume of the scatterers. In spite of the conceptual simplicity of this approximation scheme, to our knowledge concrete applications have not been worked out.

We plan to come back to this problem in further work.

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